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## NON-EUCLIDEAN GEOMETRY.

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By GEORGE BRUCE HALSTED.

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In writing of "The Wonderful Century," Alfred Russel Wallace says of all time before the seventeenth century: "Then, going backward, we can find nothing of the first rank except Euclid's wonderful system of geometry, perhaps the most remarkable mental product of the earliest civilizations."

But of late all men of science and intelligent teachers have been hearing more and more of non-Euclidean geometry, and are naturally anxious to know how these new doctrines are related to the traditional geometry which they were taught and perhaps now are teaching.

The new departure is absolutely epoch-making, but fortunately it has intensified admiration for that imperishable model, already in dim antiquity a classic, the immortal *Elements* of Euclid.

But without assumptions nothing can be proved, and Euclid stated his assumptions with the most painstaking candor. He would have smiled at the suggestion that he could ever claim for his conclusions any other truth than perfect deduction from assumed hypotheses.

And so his system is forever safe. Each one of his axioms may turn out to be inconsistent with external reality; each of his fundamental assumptions may be replaced in our final explanation of the space in which we live and move; in reference to our space, all his theorems may be shown to be only approximations; and yet his work will remain a perfect piece of pure mathematics, the exact, eternal geometry of Euclidean space.

For two thousand years no one ever doubted the truth of any one of this

set of axioms, far the most influential in the intellectual history of the world, put together by Euclid in Egypt, but really owing nothing to the Egyptian race, nothing to the boasted lore of Egypt's priests.

The Papyrus of the Rhind, belonging to the British Museum, but given to the world by the erudition of a German Egyptologist, Eisenlohr, and a German historian of mathematics, M. Cantor, gives us more knowledge of the state of mathematics in ancient Egypt than all else previously accessible to the modern world. Its whole testimony confirms with overwhelming force the position that geometry as a science, strict and self-conscious deductive reasoning, was created by the subtle intellect of the same race whose bloom in art still overawes us in the Venus of Milo, the Apollo Belvidere, the Laocoön. But though for twenty centuries the truth of the axioms of the Greek geometer remained unquestioned, there was one of them of which the axiomatic character was doubted even from far antiquity. Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek as the Parthenon. But among Euclid's assumptions is one differing from the others in prolixity, whose place fluctuates in the manuscripts.

Peyrard, on the authority of the Vatican MS., puts it among the postulates, and it is often called the parallel postulate. Heiberg, whose edition of the Greek text is the latest and the best (Leipzig, 1883-1888), gives it as the fifth postulate.

James Williamson, who published the closest translation of the Euclid we have in English, indicating, by the use of italics, the words not in the original, gives this assumption as eleventh among the Common Notions.

Bolyai speaks of it as Euclid's Axiom XI. Todhunter has it as twelfth of the axioms. Clavius (1574) gives it as axiom 13. The Harper Euclid separates it by forty-eight pages from the other axioms.

It is not used in the first twenty-eight propositions of Euclid. Moreover, when at length used, it appears as the inverse of a proposition already demonstrated, the seventeenth, and is only needed to prove the inverse of another proposition already demonstrated, the twenty-seventh.

Geminus of Rhodes (about 70 B. C.) speaks of it as needing proof. The astronomer Ptolemy (A. D. 87-165) tried his hand at proving it. The great Lambert expressly says that Proklus demanded a proof of the assumption because when inverted it is demonstrable. The Arab Nasir-Eddin (1201-1274) tried to demonstrate it.

No one had a doubt of the necessary external reality and exact applicability of the assumption. Until the present century the Euclidean geometry was supposed to be the only possible form of space-science; that is, the space analyzed in Euclid's axioms was supposed to be the only non-contradictory sort of space. But could not this assumption be deduced from the other assumptions and the twenty-eight propositions already proved by Euclid without it? Euclid demonstrated things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third.

Yet after he has finished his demonstration, that straight lines making with the transversal equal alternate angles are parallel, in order to prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy assumption thus translated by Williamson (Oxford, 1781):

“11. And if a straight line meeting two straight lines make those angles which are inward and upon the same side of it less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles.”

As Staeckel says, “it requires a certain courage to declare such a requirement, alongside the other exceedingly simple assumptions and postulates.”

In the brilliant new light given by Bolyai and Lobachevski we now see that Euclid understood the crucial character of the question of parallels.

There are now for us no better proofs of the depth and systematic coherence of Euclid's masterpiece than the very things which, their cause unappreciated, seemed the most noticeable blots on his work.

Sir Henry Savile, in his *Praelectiones* on Euclid, Oxford, 1621, p. 140, says: “In pulcherrimo Geometriae corpore duo sunt naevi, duae labe . . .” etc., and these two blemishes are the theory of parallels and the doctrine of proportion; the very points in the elements which now arouse our wondering admiration.

But down to our very nineteenth century an ever renewing stream of mathematicians tried to wash away the first of these supposed stains from the most beautiful body of geometry: First, those in which is taken a new definition of parallels. Second, those in which is taken a new axiom different from Euclid's. Third, the largest and most desperate class of attempts, namely those which strive to deduce the theory of parallels from reasonings about the nature of the straight line and plane angle. Hundreds of mathematicians tried at this. All failed. That eminent man Legendre was trying at this, and continually failing at it, throughout his very long life. Thus the experience of two thousand years went to show that here some assumption was indispensable. Every species of effort was made to avoid or elude it, but without success. From a letter of Gauss we see that in 1799 he was still trying to prove that Euclid's is the only non-contradicting system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven.

Yet even in 1831 the acute logician De Morgan accepted and reproduced a wholly fallacious proof of Euclid's assumption, recently republished, Chicago, 1898. A like pseudo-proof published in Crelle's Journal (1834) deceived even our well known Professor W. W. Johnson, who translated and published it in the *Analyst* (Vol. III, 1876, p. 103), saying, “this demonstration seems to have been generally overlooked by writers of geometrical text-books, though apparently exactly what was needed to put the theory upon a perfectly sound basis.”

The most interesting, and perhaps the most extended of such attempted proofs was by the Italian Jesuit Saccheri, born the fifth of September, 1667, who

joined the Society of Jesus at Genoa on the twenty-fourth of March, 1685. He became teacher of grammar in the Jesuit "Collegio di Brera," where the teacher of mathematics was Tommaso Ceva, a brother of the well-known mathematician Giovanni Ceva (1648-1737, who published in 1678 at Milan a work containing the theorem now known by his name.

Saccheri was in close scientific communion with both brothers and received his inspiration from them. He used Ceva's ingenious methods in his first published work, 1693, solutions of six geometric problems proposed by Count Roger Ventimiglia. His attempt at proving the parallel-postulate is his last work.

"Euclid vindicated from every fleck," which received the "Imprimatur" of the inquisition the thirteenth of July, 1733, that of the Provincial of the Jesuits the sixteenth of August, 1733. Saccheri died the twenty-fifth of October, 1733. All preceding attempts were alike in trying to give a direct positive proof of the postulate; all were alike in their assumption open or hidden, conscious or unconscious, of an equivalent postulate.

Saccheri tries a wholly new way, and thus his book marks an epoch. He never doubted the absolute necessary truth of Euclid's postulate, and so he thinks that the two alternatives, possible if it be taken as not true, must each lead to some contradiction, to some absurdity. He tries the *reductio ad absurdum*. Ninety years later, 1823, Bolyai János reached the astounding conviction that these alternatives lead not to any contradiction but to the "science absolute of space," a generalization of Euclid's universe. In a letter dated the third of November, 1823, written in the Magyar language, and fortunately preserved for us at Maras Vásárhely in Hungary, Bolyai János writes to his father Bolyai Farkas: "I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you yourself will acknowledge it. Now I cannot say more, only so much: *That from nothing I have created another wholly new world.*"

Suppose we take a few steps into this new universe on the path which opened before Saccheri without his ever suspecting whither it led.

1. If two points determine a line it is called a straight.
2. If two straights make with a transversal equal alternate angles, they have a common perpendicular.
3. A piece of a straight is called a sect.
4. If two equal coplanar sects are erected perpendicular to a straight, if they do not meet, then the sect joining their extremities makes equal angles with them and is bisected by a perpendicular erected midway between their feet. [Proved by folding the figure over, along the third perpendicular.]
5. Considering figures where the right angles made by the equal perpendiculars may be said to be not alternate, and where no two perpendiculars to the same straight meet, the equal angles made with the joining sect at the extremities of the two equal perpendiculars are either right angles, acute angles, or obtuse angles. Distinguish the three cases as hypothesis of right, hypothesis of acute, hypothesis of obtuse.

6. According to these three hypotheses respectively, the join of the extremities of the equal perpendiculars is equal to, greater than, or less than the join of their feet. [Saccheri, Prop. III. Translated by Halsted in the American Mathematical Monthly.

7. Inversely, according as the join of the extremities is equal to, or less than, or greater than the join of the feet, the equal angles will be right, or obtuse, or acute. [S. P. IV.]

8. Corollary. In every quadrilateral containing three right angles and one obtuse, or acute, the sides adjacent to this oblique angle are less than the opposite sides, if this angle is obtuse, but greater if it is acute.

9. The hypothesis of right, if even in a single case it is true, always in every case it alone is true. [S. P. V.]

10. Assuming the principle of continuity, and referring only to figures where no two perpendiculars to the same straight meet; The hypothesis of obtuse, if even in a single case it is true, always in every case it alone is true. [S. P. VI.]

11. With like limitation; The hypothesis of acute, if even in a single case it is true, always in every case it alone is true. [S. P. VII.]

12. The sum of the angles of the rectilined triangle is a straight angle in the hypothesis of right, is greater than a straight angle in the hypothesis of obtuse, is less than a straight angle in the hypothesis of acute. [S. P. IX.]

13. The excess of a triangle is the excess of the sum of its angles over a straight angle. The deficiency of a triangle is what its angle-sum lacks of being a straight angle.

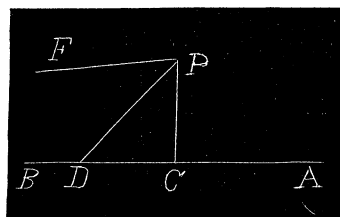
14. Two triangles having the same excess or deficiency are equivalent.

15. Even with the assumption that two straights cannot intersect in two points, the three hypotheses give rise to three perfect systems of geometry, the hypothesis of right to Euclid, the hypothesis of acute to Bolyai-Lobachevski, the hypothesis of obtuse to Riemann.

16. In the hypothesis of acute the straight is infinite. Two coplanar straights perpendicular to a third diverge on either side of their common perpendicular. The angle-sum of any rectilineal triangle is less than a straight angle.

17. In Euclid and Bolyai, parallels are straights on a common point at infinity.

18. In Bolyai from any drop point  $PPC$  a perpendicular to a given straight  $AB$ . If  $D$  move off indefinitely on the ray  $CB$  the sect  $PD$  will approach as limit  $PF$  copunctal with  $AB$  at infinity.  $PF$  is said to be at  $P$  the parallel to  $AB$  toward  $B$ .  $PF$  makes with  $PC$  an angle  $CPF$  which is called the angle of parallelism for the perpendicular  $PC$ . It is less than a right angle by an amount which is the limit of the deficiency of the triangle  $PCD$ . On the other side of  $PC$  an equal angle of parallelism gives us the parallel at  $P$  to  $BA$  toward  $A$ .



Thus at any point there are two parallels to a straight. A straight has two distinct separate points at infinity.

Straights through  $P$  which make with  $PC$  an angle greater than the angle of parallelism and less than its supplement do not meet the straight  $AB$  at all, not even at infinity.

19. A straight maintains its parallelism at all its points. [Lobachevski, Geometrical Researches on the Theory of Parallels, Translated by Halsted, §17.]

20. If one straight is parallel to a second, the second is parallel to the first. [L. §18.]

21. Two straights parallel to a third toward the same part are parallel to each other. [L. §25.]

22. Parallels continually approach each other. [L. §24.]

23. The perpendiculars erected at the middle points of the sides of a triangle are all parallel if two are parallel. [L. §30.]

24. If the foot of a perpendicular slides on a straight, its extremity describes a curve called an equi-distant curve or an equidistancial. An equidistancial will slide on its trace.

25. A circle with infinite radius is not a straight but a curve called the boundary curve, which is a plane curve for which all perpendiculars erected at the mid-points of chords are parallel. [L. §31.] It is an equidistancial whose base line is infinitely removed.

Circles, boundary-curves, equidistancials cut at right angles a system of copunctal straights, of parallel straights, of perpendiculars to a straight, respectively.

Three points determine one of these curves; that is through any three points not costraight will pass either a circle, a boundary-curve, or an equidistancial, and only one such curve.

Any triangle may be inscribed in one and only one of these curves.

26. Boundary-surface we call that surface generated by the revolution of a boundary-curve about one of its axes. Principal plane we call each plane passed through an axis of the boundary-surface.

Every principal plane cuts the boundary-surface in a boundary-curve. Any other plane cuts the boundary surface in a circle.

Boundary-triangles whose sides are arcs of the boundary-curve on the boundary-surface have the same interdependence of angles and sides and the same angle sum as rectilinear triangles in Euclid. Geometry on the boundary surface is the same as the ordinary Euclidean plane geometry. [L. §34.]

27. Triangles on an equidistant-surface are similar to their projections on the base plane; that is, they have the same angles and their sides are proportional.

28. In the hypothesis of obtuse, a straight is of finite size, and returns into itself. This size is the same for all straights. Any two straights can be made to coincide. Two straights always intersect. Two straights perpendicular to a third intersect at a point half a straight from the third either way.

29. A straight in the hypothesis of obtuse does not divide the plane into hemiplanes. Starting from the point of intersection of two straights and passing along one of them over a certain finite sect, we come again to the intersection without having crossed the other straight.

This sect is the whole straight, and so a straight has not really two sides. There is one point through which pass all the coplanar perpendiculars to a given straight. It is called the pole of that straight, and the straight is its polar.

A pole is half a straight from its polar. A polar is the locus of coplanar points half a straight from its pole. Therefore if the pole of one straight lies on another straight, the pole of this second straight is on the first straight.

The cross of two straights is the pole of the join of their poles. The equidistantial is a circle with center at the pole of its basal straight.

Three straights each perpendicular to the other two form a tri-rectangular triangle. It is self-polar, each vertex being the pole of the opposite side.

30. In the hypothesis of obtuse, any two straights enclose a plane figure, a digon. Two digons are congruent if their angles are equal.

31. In the hypothesis of obtuse, all perpendiculars to a plane meet at a point, the pole of the plane. It is the center of a system of spheres of which the plane is a limiting form when the radius becomes equal to half a straight.

Figures on a plane can be projected from similar figures on any sphere which has the pole of the plane for center. They have equal angles and corresponding sides in a constant ratio depending only on the radius of the sphere.

Geometry on a plane is therefore like two-dimensional spherics, but the plane corresponds to only a hemisphere.

The plane is unbounded but not infinite. It is finite in extent. The universe is unbounded but not infinite. It is finite in extent, or content, or volume.

Now of these three possible geometries of uniform space, Euclid's has the unexpected disadvantage that it can never be proved to be the system actual in our external physical world. To establish Euclid, it would be necessary to show that the angle-sum of a triangle is *exactly* a straight angle; and no measurements can ever reach exactitude.

To prove one of the others, we have only to show that the sum of the angles of some triangle is *less than*, or *greater than* a straight angle, which may conceivably be done even by inexact measurements.

What changes ought to be made in teaching elementary geometry in consequence of these later discoveries and the principles of the non-Euclidean geometries?

We are given a new criterion for questions of method, of exposition. For example, surface spherics attains a new importance. When properly founded and expounded, pure spherics, two-dimensional spherics, while giving all the old results and laying the foundation for spherical trigonometry, gives also a picture of the planimetric part of Riemann's geometry, and becomes a touchstone for detecting the fallacies and assumptions in the many pseudo-proofs accepted in the past, such as attempts to found parallelism on direction, attempts to prove all right angles equal, etc.



As another example, we see a new stress laid on the incalculable advantages, educational and scientific, of Euclid's procedure in deducing from three assumed constructions every other construction before he uses it in any demonstration.

The glib method of *supposed* solutions to all desired problems, of hypothetical constructions, is now seen in its deformity and danger. Euclid says, under the heading "Postulates:—"

"I. It is assumed, that a straight line may be drawn from any one point to any other point.

"II. And that a terminated straight line [a sect] may be produced in a straight line continually.

"III. And that a circle may be described with any center and radius."

From these Euclid rigidly deduces every problem of construction he wishes to use. Says Helmholtz: "In drawing any subsidiary line for the sake of his demonstration, the well-trained geometer asks always if it is possible to draw such a line. It is notorious that problems of construction play an essential part in the system of geometry.

At first sight these appear to be practical operations, introduced for the training of learners; but in reality they have the force of existential propositions. They declare that points, straight lines, or circles, such as the problem requires to be constructed, are possible under all conditions, or they determine any exceptions that there may be."

Euclid's first three propositions are problems.

The most popular American geometry, Wentworth's, (1899), puts Euclid's two first postulates on page 8, and the third postulate a whole book later, and then never has a single problem of construction until page 112, where he says: "Hitherto we have supposed the figures constructed."

Meantime, on page 88, he gives as a "theorem:—" "Through three points not in a straight line one circumference, and only one, can be drawn."

He gives as his "Proof. Draw the chords  $AB$  and  $BC$ . At the middle points of  $AB$  and  $BC$  suppose perpendiculars erected. These perpendiculars will intersect at some point  $O$ , since  $AB$  and  $BC$  are not in the same straight line."

Now the tremendous existential import of the problem, to draw a circle through three non-costraight points, will be recognized when I say that in general it is not possible. In the Lobachevski geometry not every triangle has its vertices concyclic. Granting that every three points must be costraight or concyclic, we could prove the parallel-postulate.

Of the possible geometries we cannot say *a priori* which shall be that of our actual space, the space in which we move.

The hereditary geometry, the Euclidean, is underivable from real experience alone, and can never be proved by experience. Euclidean space is, at least in part, a creation of the human mind. Its adequacy as a subjective form for experience has not yet been disproved.

It can never be proved.

The realities which with the aid of our subjective space form we understand under motion and position, may, with the coming of more accurate experience refuse to fit in that form. Our mathematical reason may decide that they would be fitted better by a non-Euclidean space form.

Comparative geometry finally overthrows that superficial method which pretends to found a logically sound exposition of geometry on "direction," undefined.

For more than twenty years Wentworth gave as his definition "A straight line is a line which has the same direction throughout its whole extent." [1877, Def. 8. 1886, p. 4; 1888, §17.]

At last he discards his aged error, and takes the definition of non-Euclidean geometry, "a straight is the line determined by two points." [1899, §§36 and 46.]

Though the Bolyai and the Riemann geometries are founded on the straight, yet to say in them of two straights that they have the same direction has no ordinary meaning, since in Riemann every two straights cross and inclose a space, while in Bolyai every two parallels continually approach each other. So as to direction, Wentworth has reformed, after twenty years in the land of Nod. But he still says, 1899, §49: "A straight line is the shortest line that can be drawn from one point to another."

Now a relation of equality or inequality between two magnitudes must have some foundation, and be capable of some intelligible test. In the traditional geometry the foundation of all proof by Euclid's method consists in establishing the congruence of magnitudes. To make the congruence evident, the geometrical figures are supposed to be applied to one another, of course without changing their form and dimensions. But since no part of a curve can be congruent to any piece of a straight, so, for example, no part of a circle can be equivalent to any sect from the definition of equivalent magnitudes as those which can be cut into pieces congruent in pairs.

In any comparison of size by congruence, we must be able to place one of the magnitudes or portions of it in complete or partial coincidence with the other. No such direct comparison can be instituted between a straight and a line no piece of which is straight.

Thus the whole of Euclid's *Elements* fails utterly to institute or prove any relation as regards size between a sect and an arc joining the same two points. The operation of measurement we cannot effect, rigorously speaking, either for curves or for curved surfaces, since the unit for length is a sect, and the unit for area, the square on that sect. In fact, however little may be the parts of a curve, they do not cease to be curved, and consequently they cannot be compared directly with a sect; just as parts of a curved surface are not directly comparable with portions of a plane.

We cannot even affirm that any ratio exists between a circle and its diameter until after we have made some extra-Euclidean and post-Euclidean assumption at least equivalent to the following:

No minor arc is less than its chord; and no arc is greater than the sum of

the tangents at its extremities. If the curve be other than a circle we assume that on it one can always take two points so near that the arc between these points is not less than its chord, nor greater than the broken line formed by the two tangents touching its extremities. Some such assumption is, in fact, necessary, but it destroys by itself the primitive idea of measuring curves with straights.

Duhamel gives the assumption the following form : The length of a curve shall be the limit toward which the length of a broken line made up of consecutive chords of that curve approaches, when the number of chords is increased in such a manner that the chords all approach zero as a limit. Thus the elevation of the length of a curve represents not at all an attempt at rectification strictly ; but it has for aim the finding of a limit to which another magnitude would approach.

In geometry one proves that as the subdivisions are increased and the sides tend toward the limit zero, the perimeter of the polygon inscribed in a circle increases, circumscribed decreases, toward the same limit, which then is assumed for the magnitude of the circle.

Therefore when Phillips and Fisher, of Yale, give as their definition of a straight [1898, p. 4, §7. Def.] "A *straight line* is a line which is the shortest path between any two of its points," they pass through and beyond Euclid's *Elements* to give us his simplest element ; they institute a comparison not only with circular arcs, but also with all curves known and unknown ; they presuppose a foreknowledge of all lines in a definition of the simplest line. Is it still needful to say this is grossly bad logic, bad pedagogy, bad mathematics ?

The same Yale geometry blunders strikingly on p. 23, where it says : "In fact, Lobatchewsky in 1829, proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from *what we know it to be through experience*. Lobatschewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, *whereas the space in which we really live* is called *Euclidean*, because Euclid (about 300 B. C.) first wrote a systematic geometry of *our space*."

The scientific doctrine of evolution postulates a world independent of man, and teaches the outcome of man from lower forms of life in accordance with wholly natural causes. In this world of evolution experience is a teacher, but man is a creator, and the mighty examiner is death.

The puppy born blind must still be able, guided by the sense of smell, to superimpose his mouth upon a source of nourishment. The little chick, responding to the stimulus of a small bright object, must be able to bring his beak into contact with the object so as to grasp and then swallow it. The springing goat that misjudges an abyss is lost.

So too with man. His ideas must in some way correspond to this independent world, or death passes upon him an adverse judgment. But it is of the very essence of the doctrine of evolution that man's metric knowledge of this

independent world, having come by gradual betterment and through imperfect instruments, for example the eye, cannot be absolute and exact.

The results of any observations are always with certain definite limitations as to exactitude and under particular conditions. Man the creator replaces these results by assumptions presumed to have absolute precision and generality, such as, for example, the so-called axioms of Euclid.

If two natural hard objects, susceptible of high polish, be ground together, their surfaces in contact may be so smoothed as to fit closely together and slide one on the other without separating. If now a third surface be ground alternately against each of these two smooth surfaces until it accurately fits both, then we say that each of the three surfaces is approximately plane, is a piece of a plane. If one such plane be made to cut through another, we say the common line where they cross is approximately a straight. The perfect, the ideal plane, is a human creation under which we seize the imperfect data of experience.

If three approximate planes on real objects be made to cut through a fourth approximate plane, then three approximate straights are formed on this fourth plane, and in general they are found to intersect, and the figure they make we may call an approximate triangle. Such triangles vary greatly in shape. But no matter what the shape, if we cut off the six ends of any two such, and place them side by side on a plane with their vertices at the same point, the six are found with a high degree of approximation just to fill up the plane about the point. If the whole angular magnitude about any point in a plane be called a perigon, then we may say that the six angles of any two approximate triangles are found to be together approximately a perigon.

Now does the exactness of this approximation to a perigon depend only on the straightness of the sides of the original two triangulars, or also upon their size?

If we know with absolute certitude that the size of the triangles has nothing to do with it, then we know something that we have no right to know according to the doctrine of evolution, something impossible for us ever to have learned evolutionally.

Yet before the epoch-making ideas of Bolyai János and Lobachevski, every one supposed we were perfectly sure that the angle-sum of an actual approximate triangle approached a straight angle with an exactness dependent only on the straightness of the sides and not at all on the size of the triangle. But if in the mechanics of the world independent of man we were absolutely certain that all therein is Euclidean and only Euclidean, then Darwinism would be disproved by the *reductio ad absurdum*.

All our measurements are finite and approximate only. The mechanics of actual bodies in what Cayley called the external space of our experience, might conceivably be shown by merely approximate measurements to be non-Euclidean, just as a body might be shown to weigh more than two grams or less than two grams, though it never could be shown to weigh precisely, absolutely two grams.

The outcome of the non-Euclidean geometry is a new freedom to explain and understand our universe and ourselves.